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Triple Solutions to Boundary Value Problems on Time Scales

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Abstract—Criteria are developed for the existence of three nonnegative solutions to nonlinear differential equations on time scales.

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1. INTRODUCTION

In this paper, we present results which guarantee the existence of three solutions to nonlinear equations on measure chains of the form

$$\begin{aligned} y^{\triangle\triangle}(t) + f(y(\sigma(t))) &= 0, \qquad \text{for } t \in [a,b], \\ \alpha\, y(a) - \beta\, y^{\triangle}(a) &= 0, \qquad \alpha > 0, \quad \beta \geq 0, \\ y^{\triangle}(\sigma(b)) &= 0. \end{aligned} \tag{1.1}$$

To understand (1.1), we need to recall some standard definitions (see [1-8] for an introduction to this subject).

DEFINITION 1.1. *Let* $\mathbf{T}$ *be a closed subset of* $\mathbf{R}$ *with the property that*

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbf{T}\} \in \mathbf{T} \qquad \text{and} \qquad \rho(t) = \sup\{\tau < t : \tau \in \mathbf{T}\} \in \mathbf{T},$$

for all $t \in \mathbf{T}$. *We assume throughout that* $\mathbf{T}$ *has the topology that it inherits from the standard topology on* $\mathbf{R}$.

Throughout this paper, $a < b$ are points in $\mathbf{T}$. Let

$$[a,b] = \{t \in \mathbf{T} : a \leq t \leq b\}.$$

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DEFINITION 1.2. *Fix $t \in \mathbf{T}$. Let $y : \mathbf{T} \to \mathbf{R}$. Then we define $y^{\triangle}(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$, there is a neighborhood U of t with*

$$\left|[y(\sigma(t)) - y(s)] - y^{\triangle}(t)\,[\sigma(t) - s]\right| < \epsilon\,|\sigma(t) - s|,$$

for all $s \in U$. We call $y^{\triangle}(t)$ the derivative of $y(t)$.

DEFINITION 1.3. *If $F^{\triangle}(t) = f(t)$, then we define the integral by*

$$\int_a^t f(\tau)\,\triangle\tau = F(t) - F(a).$$

To show the existence of three nonnegative solutions to (1.1), we will use the Leggett Williams Fixed-Point Theorem [9,10]. Recently, [9–13] this fixed-point theorem has been used to establish results for differential and difference equations. In this paper, we provide a new existence result for differential equations on time scales. In fact, our result is new when $\mathbf{T} = \mathbf{R}$ (the differential equation case) and when $\mathbf{T} = \mathbf{Z}$ (the discrete case).

For the remainder of this section, we gather together some results from the literature that will be needed in Section 2. The Green's function for the boundary value problem

$$\begin{aligned} -y^{\triangle\triangle}(t) &= 0, && \text{for } t \in [a,b],\\ \alpha\,y(a) - \beta\,y^{\triangle}(a) &= 0, && \alpha > 0, \quad \beta \ge 0,\\ y^{\triangle}(\sigma(b)) &= 0 \end{aligned} \tag{1.2}$$

is given by [5,6],

$$G(t,s) = \begin{cases} \dfrac{\alpha\,(t-a)+\beta}{\alpha}, & t \le s,\\[2mm] \dfrac{\alpha\,(\sigma(s)-a)+\beta}{\alpha}, & \sigma(s) \le t. \end{cases}$$

It is easy to check since

$$\frac{G(t,s)}{G(\sigma(s),s)} = \begin{cases} \dfrac{\alpha\,(t-a)+\beta}{\alpha\,(\sigma(s)-a)+\beta}, & t \le s,\\[2mm] 1, & \sigma(s) \le t, \end{cases}$$

that

$$0 \le G(t,s) \le G(\sigma(s),s), \qquad \text{for } t \in [a,\sigma^2(b)] \quad \text{and} \quad s \in [a,\sigma(b)] \tag{1.3}$$

and

$$G(t,s) \ge k\;G(\sigma(s),s), \qquad \text{for } t \in \left[\frac{\sigma(b)+3\,a}{4},\,\sigma^2(b)\right] \quad \text{and} \quad s \in [a,\sigma(b)]; \tag{1.4}$$

here

$$k = \frac{\alpha\,(\sigma(b)-a)+4\,\beta}{4\,\alpha\,(\sigma^2(b)-a)+4\,\beta}. \tag{1.5}$$

REMARK 1.1. We could replace $(\sigma(b)+3\,a)/4$ in (1.4) by any $\mu \in \mathbf{R}$ with $a < \mu < \sigma^2(b)$ provided k in (1.5) is adjusted to $(\alpha\,(\mu - a)+\beta)/(\alpha\,(\sigma^2(b)-a)+\beta)$.

$E = (E, \|\,.\,\|)$ is a Banach space and $C \subset E$ a cone. By a concave nonnegative continuous functional ψ on C, we mean a continuous mapping $\psi : C \to [0,\infty)$ with

$$\psi(\,\lambda\,x + (1-\lambda)\,y\,) \ge \lambda\,\psi(x) + (1-\lambda)\,\psi(y), \qquad \text{for all } x,\, y \in C \quad \text{and all} \quad \lambda \in [0,1].$$

Let K, L, $r > 0$ be constants with C and ψ as defined above. Let

$$C_K = \{y \in C : \|y\| < K\} \qquad \text{and} \qquad C(\psi, r, L) = \{y \in C : \psi(y) \ge r \text{ and } \|y\| \le L\}.$$

We now state the Leggett Williams Fixed-Point Theorem [9,10].

THEOREM 1.1. *Let $E = (E, \|\,.\,\|)$ be a Banach space, $C \subset E$ a cone of E and $R > 0$ a constant. Suppose there exists a concave nonnegative continuous functional ψ on C with $\psi(y) \le \|y\|$ for $y \in \overline{C_R}$ and let $A : \overline{C_R} \to \overline{C_R}$ be a continuous, compact map. Assume there are numbers r, L, and K with $0 < r < L < K \le R$ such that*

(H1) $\{y \in C(\psi, L, K) : \psi(y) > L\} \neq \emptyset$ *and* $\psi(A\,y) > L$ *for all* $y \in C(\psi, L, K)$;
(H2) $\|A\,y\| < r$ *for all* $y \in \overline{C_r}$;
(H3) $\psi(A\,y) > L$ *for all* $y \in C(\psi, L, R)$ *with* $\|A\,y\| > K$.

Then A has at least three fixed points y_1, y_2, and y_3 in $\overline{C_R}$. Furthermore, we have

$$y_1 \in C_r, \quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\} \quad \text{and} \quad y_3 \in \overline{C_R} \setminus \left(C(\psi, L, R) \cup \overline{C_r}\right).$$

2. EXISTENCE

We use Theorem 1.1 to establish the existence of three nonnegative solutions to (1.1). By $C[a, \sigma^2(b)]$, we mean the Banach space of continuous functions $y : [a, \sigma^2(b)] \to \mathbf{R}$ equipped with the norm $|y|_0 = \sup_{t \in [a, \sigma^2(b)]} |y(t)|$.

The following conditions will be assumed:

$$f : [0, \infty) \to [0, \infty) \text{ is continuous and nondecreasing} \tag{2.1}$$

$$\exists\, r > 0 \text{ with } f(r) \sup_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s) \,\Delta\, s < r \tag{2.2}$$

$$\begin{gathered} \left[a, \sigma^2(b)\right] \text{ is such that } \eta = \min\left\{\tau \in \mathbf{T} : \tau \ge \frac{\sigma(b) + 3\,a}{4}\right\} \\ \text{exists and satisfies } \frac{\sigma(b) + 3\,a}{4} \le \eta < \sigma(b) \end{gathered} \tag{2.3}$$

$$\exists\, L > r \text{ with } f(L) \min_{t \in [\eta, \sigma^2(b)]} \int_\eta^{\sigma(b)} G(t, s) \,\Delta\, s > L \tag{2.4}$$

and

$$\exists\, R \ge L\,k^{-1} \text{ with } f(R) \sup_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s) \,\Delta\, s \le R; \tag{2.5}$$

here k is as in (1.5).

THEOREM 2.1. *Suppose (2.1)–(2.5) hold. Then (1.1) has three nonnegative solution y_1, y_2, and y_3 in $C[a, \sigma^2(b)]$ with $y_i^{\Delta\Delta} \in C_{rd}[a, b]$ (note $y_i^\sigma = y_i \circ \sigma$ is rd-continuous on $[a, b]$) for $i = 1, 2, 3$ and*

$$|y_1|_0 < r, \quad y_2(t) > L, \qquad \text{for } t \in \left[\eta, \sigma^2(b)\right] \text{ and } |y_3|_0 > r \text{ with } \min_{t \in [\eta, \sigma^2(b)]} y_3(t) < L.$$

REMARK 2.1. Note $C_{rd}[a, b]$ is the space of rd-continuous functions [1,8] on $[a, b]$.

REMARK 2.2. Notice $(\sigma(b) + 3\,a)/4$ could be replaced by any $\mu \in \mathbf{R}$, $a < \mu < \sigma(b)$ (see Remark 1.1) provided k in (1.5) is adjusted to $(\alpha\,(\mu - a) + \beta)/(\alpha\,(\sigma^2(b) - a) + \beta)$.

PROOF. Let

$$E = \left(C\left[a, \sigma^2(b)\right], |\cdot|_0\right) \quad \text{and} \quad C = \left\{y \in C\left[a, \sigma^2(b)\right] : y(t) \ge 0, \ \text{ for } t \in \left[a, \sigma^2(b)\right]\right\}.$$

Now let $A : C \to C$ be defined by

$$A\,y(t) = \int_a^{\sigma(b)} G(t, s)\, f(y(\sigma(s))) \,\Delta\, s, \qquad \text{for } t \in \left[a, \sigma^2(b)\right]; \tag{2.6}$$

here $y \in C$. It is immediate from the ideas in [3] that

$$A : C \to C \text{ is continuous and completely continuous.}$$

For $y \in C$, let

$$\psi(y) = \min_{t \in [\eta, \sigma^2(b)]} y(t).$$

Next choose and *fix* K so that

$$L\,k^{-1} \le K \le R; \tag{2.7}$$

this is possible since $R \ge L\,k^{-1}$. Let

$$C_r = \{y \in C : |y|_0 < r\}, \qquad C_R = \{y \in C : |y|_0 < R\}$$

and

$C(\psi, L, K) = \{y \in C : \psi(y) \ge L \text{ and } |y|_0 \le K\}$, $C(\psi, L, R) = \{y \in C : \psi(y) \ge L \text{ and } |y|_0 \le R\}$.

We first show condition (H2) of Theorem 1.1 holds. If $y \in \overline{C_r}$, then (2.6) and [8, Theorem 1.4.3 (iii)] guarantee that

$$\begin{aligned} A\,y(t) &= \int_a^{\sigma(b)} G(t,s)\,f(y(\sigma(s)))\,\triangle\,s \le f(|y|_0) \int_a^{\sigma(b)} G(t,s)\,\triangle\,s \\ &\le f(r) \sup_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t,s)\,\triangle\,s, \end{aligned}$$

for $t \in [a, \sigma^2(b)]$. This together with (2.2) yields

$$|A\,y|_0 \le f(r) \sup_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t,s)\,\triangle\,s < r.$$

Thus, condition (H2) of Theorem 1.1 holds. Similarly, we have $A : \overline{C_R} \to \overline{C_R}$.

Next, we show condition (H1) of Theorem 1.1 holds. First, notice if

$$u(t) = \frac{L+K}{2}, \qquad \text{for } t \in \left[a, \sigma^2(b)\right],$$

then $u \in \{y \in C(\psi, L, K) : \psi(y) > L\}$. Also if $y \in C(\psi, L, K)$, then $\psi(y) = \min_{t \in [\eta, \sigma^2(b)]} y(t) \ge L$ and $|y|_0 \le K$, so $y(s) \in [L, K]$ for $s \in [\eta, \sigma^2(b)]$ and as a result

$$y(\sigma(x)) \in [L, K], \qquad \text{for } x \in [\eta, \sigma(b)]. \tag{2.8}$$

This together with (2.4) and [8 (Theorem 1.4.3 (iii))] yields

$$\begin{aligned} \psi(A\,y) &= \min_{t \in [\eta, \sigma^2(b)]} \int_a^{\sigma(b)} G(t,s)\,f(y(\sigma(s)))\,\triangle\,s \ge \min_{t \in [\eta, \sigma^2(b)]} \int_\eta^{\sigma(b)} G(t,s)\,f(y(\sigma(s)))\,\triangle\,s \\ &\ge f(L) \min_{t \in [\eta, \sigma^2(b)]} \int_\eta^{\sigma(b)} G(t,s)\,\triangle\,s > L, \end{aligned}$$

so condition (H1) of Theorem 1.1 is satisfied. Finally, to see that (H3) of Theorem 1.1 holds, let $y \in C(\psi, L, R)$ with $|A\,y|_0 > K$. First, notice (1.3) and [8, Theorem 1.4.3 (iii)] imply

$$|A\,y|_0 \le \int_a^{\sigma(b)} G(\sigma(s), s)\,f(y(\sigma(s)))\,\triangle\,s$$

and this together with (1.4), [8, Theorem 1.4.3 (iii)], and (2.7) yields

$$\begin{aligned} \psi(A\,y) &= \min_{t \in [\eta, \sigma^2(b)]} \int_a^{\sigma(b)} G(t,s)\,f(y(\sigma(s)))\,\triangle\,s \\ &\ge \int_a^{\sigma(b)} k\,G(\sigma(s), s)\,f(y(\sigma(s)))\,\triangle\,s \\ &\ge k\,|A\,y|_0 > k\,K \ge L. \end{aligned}$$

Thus, condition (H3) of Theorem 1.1 holds. Now apply Theorem 1.1. ∎

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